

Combinatorial Derivation

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Abstract

Let G be a group, \mathcal{P}_G be the family of all subsets of G . For a subset $A \subseteq G$, we put $\Delta(A) = \{g \in G : |gA \cap A| = \infty\}$. The mapping $\Delta : \mathcal{P}_G \rightarrow \mathcal{P}_G$, $A \mapsto \Delta(A)$, is called a combinatorial derivation and can be considered as an analogue of the topological derivation $d : \mathcal{P}_X \rightarrow \mathcal{P}_X$, $A \mapsto A^d$, where X is a topological space and A^d is the set of all limit points of A . Content: elementary properties, thin and almost thin subsets, partitions, inverse construction and Δ -trajectories, Δ and d .

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Let G be a group with the identity e , \mathcal{P}_G be the family of all subsets of G . For a subset A of G , we denote

$$\Delta(A) = \{g \in G : |gA \cap A| = \infty\},$$

observe that $\Delta(A) \subseteq AA^{-1}$, and say that the mapping

$$\Delta : \mathcal{P}_G \rightarrow \mathcal{P}_G, A \mapsto \Delta(A)$$

is a *combinatorial derivation*.

In this paper, on one hand, we analyze from the Δ -point of view a series of results from Subset Combinatorics of Groups (see the survey [9]), and point out some directions for further progress. On the other hand, the Δ -operation is interesting and intriguing by its own sake. In contrast to the trajectory $A \rightarrow AA^{-1} \rightarrow (AA^{-1})(AA^{-1}) \rightarrow \dots$, the Δ -trajectory $A \rightarrow \Delta(A) \rightarrow \Delta^2(A) \rightarrow \dots$ of a subset A of G could be surprisingly complicated: stabilizing, increasing, decreasing, periodic or chaotic. For a symmetric subset A of G with $e \in A$, there exists a subset $X \subseteq G$ such that

$\Delta(X) = A$. We conclude the paper by demonstrating how Δ and a topological derivation d arise from some unified ultrafilter construction.

We note also that $\Delta(A)$ may be considered as some infinite version of the symmetry sets well-known in Additive Combinatorics [13, p. 84]. Given a finite subset A of an Abelian group G and $\alpha \geq 0$, the symmetry set $Sym_\alpha(A)$ is defined by

$$Sym_\alpha(A) = \{g \in G : |A \cap (A + g)| \geq \alpha|A|\}.$$

1 Elementary properties

1.1. $(\Delta(A))^{-1} = \Delta(A)$, $\Delta(A) \subseteq AA^{-1}$.

1.2. $\Delta(A) = \emptyset \Leftrightarrow e \notin \Delta(A) \Leftrightarrow A$ is finite.

1.3. For subsets A, B of G , we let

$$\Delta(A, B) = \{g \in G : |gA \cap B| = \infty\}$$

and note that

$$\Delta(A \cup B) = \Delta(A) \cup \Delta(B) \cup \Delta(A, B) \cup \Delta(B, A),$$

$$\Delta(A \cap B) \subseteq \Delta(A) \cap \Delta(B)$$

.

1.4. If F is a finite subset of G then

$$\Delta(FA) = F\Delta(A)F^{-1}.$$

1.5. If A is an infinite subgroup then $A = \Delta(A)$ but the converse statement does not hold, see **5.4**.

2 Thin and almost thin subsets

A subset A of a group G is said to be [8]:

- *thin* if either A is finite or $\Delta(A) = \{e\}$;
- *almost thin* if $\Delta(A)$ is finite;
- *k-thin* ($k \in \mathbb{N}$) if $|gA \cap A| \leq k$ for each $g \in G \setminus \{e\}$;

- *sparse* if, for every infinite subset $X \subseteq G$, there exists a non-empty finite subset $F \subset X$ such that $\bigcap_{g \in F} gA$ is infinite;
- *k-sparse* ($k \in \mathbb{N}$) if, for every infinite subset $X \subseteq G$, there exists a subset $F \subset X$ such that $|F| \leq k$ and $\bigcap_{g \in F} gA$ is finite.

The following statements are from [8].

2.1. Every almost thin subset A of a group G can be partitioned in $3^{|\Delta(A)|-1}$ thin subsets. If G has no elements of odd order, then A can be partitioned in $2^{|\Delta A|-1}$ thin subsets.

2.2. A subset A of a group G is 2-sparse if and only if $X^{-1}X \not\subseteq \Delta(A)$ for every infinite subset X of G .

2.3. For every countable thin subset A of a group G , there is a thin subset B such that $A \cup B$ is 2-sparse but not almost thin.

2.4. Suppose that a group G is either torsion-free or, for every $n \in \mathbb{N}$, there exists a finite subgroup H_n of G such that $|H_n| > n$. Then there exists a 2-sparse subset of G which cannot be partitioned in finitely many thin subsets.

By **2.2**, every almost thin subset is 2-sparse. By **2.3**, **2.4**, the class of 2-sparse subsets is wider than the class of almost thin subsets. By **2.3**, a union of two thin subsets needs not to be almost thin. By **1.3**, a union $A_1 \cup \dots \cup A_n$ of almost thin subset is almost thin if and only if $\Delta(A_i, A_j)$ is finite for all $i, j \in \{1, \dots, n\}$. By **1.4**, if A is almost thin and K is finite then KA is almost thin.

The following statements are from [7].

2.5. For every infinite group G , there exists a 2-thin subset such that $G = XX^{-1} \cup X^{-1}X$.

2.6. For every infinite group G , there exists a 4-thin subset such that $G = XX^{-1}$.

Since $\Delta(X) = \{e\}$ for each infinite thin subset of G , **2.6** gives us X with $\Delta(X) = \{e\}$ and $XX^{-1} = G$.

3 Large and small subsets

A subset A of a group G is called [8]:

- *large* if there exists a finite subset F of G such that $G = FA$;
- Δ -*large* if $\Delta(A)$ is large;
- *small* if $(G \setminus A) \cap L$ is large for each large subset L of G ;
- *P-small* if there exists an injective sequence $(g_n)_{n \in \omega}$ in G such that the subsets $\{g_n A : n \in \omega\}$ are pairwise disjoint;
- *almost P-small* if there exists an injective sequence $(g_n)_{n \in \omega}$ in G such that the family $\{g_n A : n \in \omega\}$ is almost disjoint, i.e. $g_n A \cap g_m A$ is finite for all distinct $n, m \in \omega$.
- *weakly P-small* if, for every $n \in \omega$, one can find distinct elements g_1, \dots, g_n of G such that the subsets $g_1 A, \dots, g_n A$ are pairwise disjoint.

3.1. Let G be a group, A is a large subset of G . We take a finite subset F of G , $F = \{g_1, \dots, g_n\}$ such that $G = FA$. Take an arbitrary $g \in G$. Then $g_i A \cap g A$ is infinite for some $i \in \{1, \dots, n\}$, so $g_i^{-1} g \in \Delta(A)$. Hence, $G = F\Delta(A)$ and A is Δ -large. By **2.6**, the converse statement is very far from being true.

If A is not small then FA is thick (see **4.3**) for some finite subset F . It follows that $\Delta(FA) = G$. By **1.4**, $\Delta(FA) = F\Delta(A)F^{-1}$, so if G is Abelian then A is Δ -large.

Question 1. *Is every nonsmall subset of an arbitrary infinite non-Abelian group G Δ -large?*

3.2. It is easy to see that A is P-small (almost P-small) if and only if there exist an infinite subset X of G such that $X^{-1}X \cap PP^{-1} = \{e\}$ ($X^{-1}X \cap \Delta(X) = \{e\}$). A is weakly P-small if and only if, for every $n \in \omega$, there exists $F \subset G$ such that $|F| = n$ and $F^{-1}F \cap PP^{-1} = \{e\}$.

3.3. By [8, Lemma 4.2], if AA^{-1} is not large then A is small and P-small. Using the inverse construction **5.3**, we can find A such that A is not Δ -large and A is not P-small.

3.4. Every infinite group G has a weakly P-small not P-small subsets [1]. Moreover, G has almost P-small not P-small subset and, if G is countable, weakly P-small not almost P-small subset. By [8], every almost P-small subset can be partitioned in two P-small subsets. If A is either almost or weakly P-small then $G \setminus \Delta(A)$ is infinite, but a subset A with infinite $G \setminus \Delta(A)$ could be large: $G = \mathbb{Z}$, $A = 2\mathbb{Z}$.

4 Partitions

4.1. Let G be a group and let $G = A_1 \cup \dots \cup A_n$ be a finite partition of G . In section 6, we show that at least one cell A_i is Δ -large, in particular, $A_i A_i^{-1}$ is large. If G is infinite amenable group and μ is a left invariant Banach measure on G , we can strengthen this statement: there exist a cell A_i and a finite subset F such that $|F| \leq n$ and $G = F\Delta(A_i)$. To verify this statement, we take A_i such that $\mu(A_i) \geq \frac{1}{n}$ and choose distinct g_1, \dots, g_m such that $\mu(g_k A_i \cap g_l A_i) = 0$ for all distinct $k, l \in \{1, \dots, m\}$, and the family $\{g_1 A_i, \dots, g_m A_i\}$ is maximal with respect to this property. Clearly, $m \leq n$. For each $g \in G$, we have $\mu(g A_i \cap g_k A_i) > 0$ for some $k \in \{1, \dots, m\}$ so $g_k^{-1} g \in \Delta(A_i)$ and $G = \{g_1, \dots, g_m\} \Delta(A_i)$.

4.2. By [10, Theorem 12.7], for every partition $A_1 \cup \dots \cup A_n$ of an arbitrary group G , there exist a cell A_i and a finite subset F of G such that $G = F A_i A_i^{-1}$ and $|F| \leq 2^{2^{n-1}-1}$. S. Slobodianiuk strengthened this statement: there are F and A_i such that $|F| \leq 2^{2^{n-1}-1}$ and $G = F\Delta(A_i)$.

It is an old unsolved problem [5, Problem 13.44] whether i and F can be chosen so that $G = F A_i A_i^{-1}$ and $|F| \leq n$, see also [10, Question 12.1].

Question 2. *Given any partition $G = A_1 \cup \dots \cup A_n$, do there exist F and A_i such that $G = F\Delta(A_i)$ and $|F| \leq 2^n$?*

4.3. A subset A of a group G is called [11]:

- *thick* if $G \setminus A$ is not large;
- *k-prethick* ($k \in \mathbb{N}$) if there exists a subset F of G such that $|F| \leq k$ and FA is thick;
- *prethick* if A is k -prethick for some $k \in \mathbb{N}$.

By [10, Theorem 11.2], if a group G is partitioned into finite number of cells, then at least one cell of the partition is prethick. We say that a partition \mathcal{P} of a group G is *k-meager* if each number of \mathcal{P} is not k -prethick. If G is either countable locally finite, or countable residually finite, or infinite Abelian, then G admits a k -meager 2-partition for each $k \in \mathbb{N}$ [11].

4.4. By [3, Theorem 5.3.2], for a group G , the following two conditions (i) and (ii) are equivalent:

- (i) for every partition $G = A \cup B$, either $G = AA^{-1}$ or $G = BB^{-1}$;
- (ii) each element of G has odd order.

If G is infinite, we can show that these conditions are equivalent to

- (iii) for every partition $G = A \cup B$, either $G = \Delta(A)$ or $G = \Delta(B)$.

5 Inverse construction and Δ -trajectories

5.1. Let G be an infinite group, $A \subseteq G$, $A = A^{-1}$, $e \in A$. We construct a subset X of G such that $\Delta(X) = A$.

First, assume that G is countable and write the elements of A in the list $\{a_n : n < \omega\}$, if A is finite then all but finitely many a_n are equal to e . We represent $G \setminus A$ as a union $G \setminus A = \bigcup_{n \in \omega} B_n$ of finite subsets such that $B_n \subseteq B_{n+1}$, $B_n^{-1} = B_n$. Then we choose inductively a sequence $(X_n)_{n \in \omega}$ of finite subsets of G ,

$$X_n = \{x_{n0}, x_{n1}, \dots, x_{nn}, a_0 x_{n0}, \dots, a_n x_{nn}\}$$

such that $X_m X_n^{-1} \cap B_n = \{e\}$ for all $m \leq n < \infty$.

After ω steps, we put $X = \bigcup_{n \in \omega} X_n$. By the construction, $\Delta(X) = A$.

If $|A| \leq \aleph_0$ but G is not countable, we take a countable subgroup H of G such that $A \subseteq H$, forget about G and find a subset $X \subseteq H$ such that $\Delta(X)$ is equal to A in H . Since $gA \cap A = \emptyset$ for each $g \in G \setminus H$, we have $\Delta(X) = A$.

At last, let $|A| > \aleph_0$. By above paragraph, we may suppose that $|A| = |G|$. We enumerate $A = \{a_\alpha : \alpha < |G|\}$ and construct inductively a sequence $(X_\alpha)_{\alpha < |G|}$ of finite subsets of G and an increasing sequence $(H_\alpha)_{\alpha < |G|}$ of subgroup of G such that if $\alpha = 0$ or α is a limit ordinal, $n \in \omega$,

$$X_{\alpha+n} = \{x_{\alpha+n,0}, x_{\alpha+n,1}, \dots, x_{\alpha+n,n}, a_\alpha x_{\alpha+n,0}, \dots, a_{\alpha+n} x_{\alpha+n,n}\},$$

$$X_{\alpha+n} \subseteq H_{\alpha+n+1} \setminus H_{\alpha+n}, \quad X_{\alpha+n} X_{\alpha+n}^{-1} \subseteq A \cup (H_{\alpha+n+1} \setminus H_{\alpha+n}).$$

After $|G|$ steps, we put $X = \bigcup_{\alpha < |G|} X_\alpha$. By the construction, $\Delta(X) = A$.

5.2. Let A_1, \dots, A_m be subsets of an infinite group G such that $G = A_1 \cup \dots \cup A_m$. By the Hindman theorem [4, Theorem 5.8], there exists $i \in \{1, \dots, m\}$ and an injective sequence $(g_n)_{n \in \omega}$ in G such that $FP(g_n)_{n \in \omega} \subseteq A_i$, where $FP(g_n)_{n \in \omega}$ is a set of all element of the form $g_{i_1} g_{i_2} \dots g_{i_l}$, $i_1 < \dots, i_k < \omega$, $k \in \omega$.

We show that there exists $X \subseteq FP(g_n)_{n \in \omega}$ such that $\Delta(X) = \{e\} \cup FP(g_n)_{n \in \omega} \cup (FP(g_n)_{n \in \omega})^{-1}$. We note that if G is countable, at each step n of the inverse construction 5.2, the elements x_{n0}, \dots, x_{nn} can be chosen from any pregiven infinite subset Y of G . We enumerate $FP(g_n)_{n \in \omega}$ in a sequence $(a_n)_{n \in \omega}$ and put $Y = \{g_n : n \in \omega\}$. Using above observation, we get the desired X .

5.3. If G is countable, we can modify the inverse construction to get X such that $\Delta(X) = A$ and $|X \cap g_1 \cap g_2 X| < \infty$ for all distinct $g_1, g_2 \in G \setminus \{e\}$, in particular, X is 3-sparse and, in particular, small.

Another modification, we can choose X such that $X \cap gX \neq \emptyset$ for each $g \in G$. If we take A not large, then we get X which is not P-small and X is not Δ -large, see **3.3**.

5.4. Let G be a countable group such that, for each $g \in G \setminus \{e\}$, the set $\sqrt{g} = \{x \in G : x^2 = g\}$ is finite. The following statements show that all possible Δ -trajectories of subsets of G can be realized.

- (Tr₁) Given any subset $X_0 \subseteq G$, $X_0 = X_0^{-1}$, $e \in X_0$, there exists a sequence $(X_n)_{n \in \omega}$ of subsets of G such that $\Delta(X_{n+1}) = X_n$ and $X_m \cap X_n = \{e\}$, $0 < m < n < \omega$.
- (Tr₂) There exists a sequence $(X_n)_{n \in \mathbb{Z}}$ of subsets of G such that $\Delta(X_n) = X_{n+1}$, $X_m \cap X_n = \{e\}$, $m, n \in \mathbb{Z}, m \neq n$.
- (Tr₃) There exists a subset A of G such that $\Delta(A) = A$ but A is not a subgroup.
- (Tr₄) There exists a subset A such that $A \supset \Delta(A) \supset \Delta^2(A) \supset \dots$
- (Tr₅) There exists a subset A such that $A \subset \Delta(A) \subset \Delta^2(A) \subset \dots$
- (Tr₆) For each natural number n , there exists a periodic Δ -trajectory X_0, \dots, X_{n-1} of length n : $X_1 = \Delta(X_0), X_2 = \Delta(X_1), \dots, X_n = \Delta(X_{n-1})$ such that $X_i \cap X_j = \{e\}$, $i < j < n$.

We use the following simple observation

- (*) if F is a finite subset of an infinite group G and $g \notin F$ then the set $\{x \in G : x^{-1}gx \notin F\}$ is infinite.

In constructions of corresponding trajectories, at each inductive step, we use a finiteness of \sqrt{g} and (*) in the following form:

- (**) if $a \in G$, F is a finite subset of G , $F \cap \{e, a^{\pm 1}\} = \emptyset$ then there exists $x \in G$ such that

$$\{x^{\pm 1}, (ax)^{\pm 1}\} \cap F = \emptyset.$$

We show how to get a 2-periodic trajectory: $X, Y, \Delta(X) = Y, \Delta(Y) = X, X \cap Y = \{e\}$. We write G as a union $G = \bigcup_{n \in \omega} F_n$ of increasing chain $\{F_n : n \in \omega\}$ of finite symmetric subsets $F_0 = \{e\}$. We put $X_0 = Y_0 = \{e\}$ and construct inductively with usage of $(^{**})$ two chains $(X_n)_{n \in \omega}, (Y_n)_{n \in \omega}$ of finite subsets of G such that, for each $n \in \omega$,

$$\begin{aligned} X_{n+1} &= \{(x(y))^{\pm 1}, (yx(y))^{\pm 1} : y \in Y_0 \cup \dots \cup Y_n\}, \\ Y_{n+1} &= \{(y(x))^{\pm 1}, (xy(x))^{\pm 1} : x \in X_0 \cup \dots \cup X_n\}, \\ (X_0 \cup \dots \cup X_n) \cap (Y_0 \cup \dots \cup Y_n) &= \{e\}, \\ X_{n+1} X_{n+1} \cap (F_{n+1} \setminus (Y_0 \cup \dots \cup Y_n)) &= \emptyset, \\ Y_{n+1} Y_{n+1} \cap (F_{n+1} \setminus (X_0 \cup \dots \cup X_n)) &= \emptyset, \\ (X_0 \cup \dots \cup X_n) X_{n+1} \cap (F_{n+1} \setminus (Y_0 \cup \dots \cup Y_n)) &= \emptyset, \\ (Y_0 \cup \dots \cup Y_n) Y_{n+1} \cap (F_{n+1} \setminus (X_0 \cup \dots \cup X_n)) &= \emptyset. \end{aligned}$$

After ω steps, we put $X = \bigcup_{n \in \omega} X_n, Y = \bigcup_{n \in \omega} Y_n$.

6 Δ and d

For a subset A of a topological space X , the subset A^d of all limit points of A is called a *derived subset*, and the mapping $d : \mathcal{P}(X) \rightarrow \mathcal{P}(X), A \rightarrow A^d$, defined on the family of $\mathcal{P}(X)$ of all subsets of X , is called a *topological derivation*, see [6, §9].

Let X be a discrete set, βX be the Stone-Ćech compactification of X . We identify βX with the set of all ultrafilters on X , X with the set of all principal ultrafilters, and denote $X^* = \beta X \setminus X$ the set of all free ultrafilters. The topology of βX can be defined by the family $\{\overline{A} : A \subseteq X\}$ as a base for open sets, $\overline{A} = \{p \in \beta X : A \in p\}$, $A^* = \overline{A} \cap X^*$. For a filter φ on X , we put $\overline{\varphi} = \{p \in \beta X : \varphi \subseteq p\}$, $\varphi^* = \overline{\varphi} \cap X^*$.

Let G be a discrete group, $p \in \beta G$. Following [2, Chapter 3], we denote

$$cl(A, p) = \{g \in G : A \in gp\}, \quad gp = \{gP : P \in p\},$$

say that $cl(A, p)$ is a *closure of A in the direction of p* , and note that

$$\Delta(A) = \bigcap_{p \in A^*} cl(A, p).$$

A topology τ on a group G is called *left invariant* if the mapping $l_g : G \rightarrow G, l_g(x) = gx$ is continuous for each $g \in G$. A group G endowed with a left

invariant topology τ is called *left topological*. We note that a left invariant topology τ on G is uniquely determined by the filter φ of neighbourhoods of the identity $e \in G$, $\overline{\varphi}$ and φ^* are the sets of all ultrafilters and all free ultrafilters of G converging to e . For a subset A of G , we have

$$A^d = \bigcap_{p \in (\tau^*)} cl(A, p),$$

and note that $A^d \subseteq \Delta(A)$ if A is a neighbourhood of e in (G, τ) .

Now we endow G with the discrete topology and, following [4, Chapter 4], extend the multiplication on G to βG . For $p, q \in \beta G$, we take $P \in p$ and, for each $g \in P$, pick some $Q_g \in q$. Then $\bigcup_{g \in P} gQ_g \in pq$ and each member of pq contains a subset of this form. With this multiplication, βG is a compact right topological semigroup. The product pq can also be defined by the rule [2, Chapter 3]:

$$A \subseteq G, A \in pq \Leftrightarrow cl(A, q) \in p.$$

If (G, τ) is left topological semigroup then $\overline{\tau}$ is a subsemigroup of βG . If an ultrafilter $p \in \overline{\tau}$ is taken from the minimal ideal $K(\overline{\tau})$ of $\overline{\tau}$, by [2, Theorem 5.0.25], there exists $P \in p$ and finite subset F of G such that $Fcl(P, p)$ is neighbourhood of e in τ . In particular, if τ indiscrete ($\tau = \{\emptyset, G\}$), $p \in K(\beta G)$ and $P \in p$ then $cl(P, p)$ is large. If G is infinite, $p \in K(\beta G)$ is free, so $cl(P, p) \subseteq \Delta(P)$ and P is Δ -large. If a group G is finitely partitioned $G = A_1 \cup \dots \cup A_n$, then some cell A_i is a member of p , hence A_i is Δ -large.

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